

Geographically Weighted Models

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1 Spatially-distributed models

Kernel functions

The bandwidth problem

2 Geographically-weighted models

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GWR example 1 – Northeast USA climate

GWR Example 2 – Georgia (USA) poverty

3 Extensions to GWR

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When considering **spatially-distributed attributes**, we can view these in two ways:

Global all spatial units are considered together

- aim: to characterize the entire population with one model (statistical summaries, regressions, ...)

Local a **geographically-compact subset** of spatial units are considered together

- aim: to see if there is **spatial heterogeneity** within the model ...
- ... and if so, at which **scale**
- general term: **Geographically-weighted** (GW) models

Global vs. local – example

- Closely related to the Modifiable Area Unit Problem (MAUP)
- Example: **Summary statistics** at different resolutions
 - MAUP: nation, state, county, town, ward ... proportion of votes per candidate
 - GW models: proportion of different soil types over the entire map vs. sub-maps; e.g., northern vs. southern Tompkins County
- Example: Empirical-statistical models example:
regression on covariates
 - MAUP: regression model of votes vs. demography
 - GW models: relation of soil properties to covariates (elevation, slope, ...)

Why build **local** models?

- **Detect** whether there is **spatial heterogeneity** in what is being studied
- **Detect** the **spatial scale** of this heterogeneity
- From these, **explain** why

Approaches to local models

Moving window re-compute summaries, regressions etc. for the observations within some *window*, i.e., restricted neighbourhood

- this neighbourhood moves across the study area

Weighted moving window same, but *weight* the observations

- closer to the window centre receive more weight than further
- requires a **kernel** function defining the weight
- function of **distance** from the centre of the window

Several possibilities:

- ① **regular tessellation**: centres of pre-defined grids
 - e.g., 10 x 10 km grid
 - result is a model, statistics etc. for each pre-defined grid
- ② **at observation points**; may be irregular
 - result is a model, statistics etc. for each observation point and its neighbourhood

- These define the weights to be given to observations within a window
- Model **form**: various forms of **distance d decay**, see next slide
- Parameter: **bandwidth h** , relation to d
- Can choose between model forms and select bandwidth by **cross-validation**, see next section
 - But often the model form is set by the knowledge of the target variable

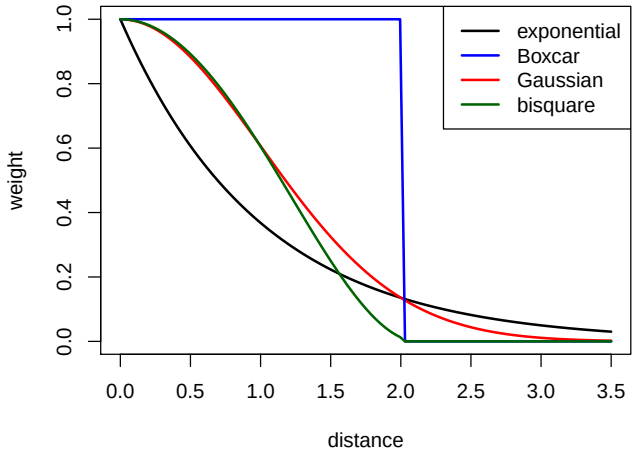
boxcar $w_{ij} = 1$ if $d_{ij} \leq h$, else $w_{ij} = 0$: unweighted within a neighbourhood

bisquare $w_{ij} = (1 - (d_{ij}^2/h^2))^2$ if $d_{ij} \leq h$, else $w_{ij} = 0$; inverse square within some neighbourhood

exponential $w_{ij} = e^{-d_{ij}/h}$; considers all the points, with exponentially decaying weight; reaches a weight of 0.5 at $d = -\log(0.5) \approx 0.693h$

Gaussian $w_{ij} = e^{-d_{ij}^2/2h^2}$; considers all the points, with exponentially decaying weight; reaches a weight of 0.5 at $d = h\sqrt{-2\log(0.5)} \approx 1.117h$

Kernel weighting functions



How “local” is local?

- Obviously, we do not want to fit too **narrowly**, because:
 - not enough sample points to reliably calibrate a model;
 - artificial local variability, *not* corresponding to the **process**.
- But we do not want to fit too **broadly**, because this would miss “true” local variability

This is the **bandwidth problem** – it should correspond to the **process** which varies locally.

- the **bandwidth** h parameter in the kernel functions determines the range of influence of points in the regression . . .
- . . . their **relative weights** is determined by the kernel function

The bandwidth can vary across the map or not:

fixed as the **distance parameter** h in the above formulations

- This corresponds to a process with a fixed dependence on distance

adaptive a **proportion** of the points to use for each local fit

- This is appropriate if points are irregularly spread – it ensures that there are enough points to calibrate the regression.
- It also mitigates edge effects with fewer points

Figure 13.2 A fixed spatial weighting function.

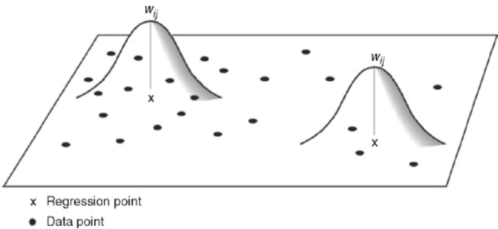
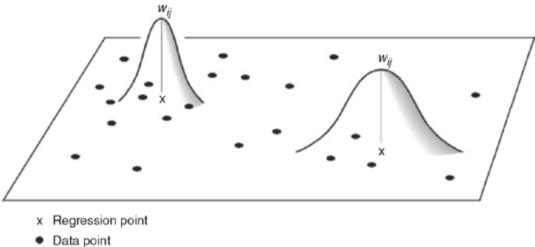


Figure 13.3 A spatially adaptive weighting function.



source: [2]

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GWR example 1 – Northeast USA climate

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3 Extensions to GWR

These have:

- any statistical **model form**;
- use a **weighted moving window**;
- a **kernel function** to define the neighbourhood;
- defined **centres**, either on on each observation point *or* a set of prediction points

Geographically-weighted regression (GWR)

- developed by Fotheringham et al. [3];
- an extension of linear or generalized linear regression;
- GWR **fits** the regression equation at **each data point** ...
 - ... based on some **neighbourhood** and ...
 - ... a **weighting scheme** (kernel function).

- GWR is appropriate if the process being modelled is **spatially non-stationary**
 - i.e., *the relation is not the same over the whole map.*
- A single global model, although representing the **overall** relation, would miss important **local variations**.
- There should be a **physical/social basis**, i.e., some reason to think there might be non-stationarity
 - why?, and over what spatial extent? (see “bandwidth problem”)
 - GWR can detect if this is the case – but careful for **artefacts** of the method: apparent variability not corresponding to the process, just to random noise

GWR gives explicit values of:

- ① the **bandwidth** within which a local regression should be fit;
 - this is determined by cross-validation
- ② the **regression coefficients** at each point
- ③ the **variability and spatial pattern** of these.

- Voting choices (e.g., percent for each political party) explained by demographic factors (income, home ownership, age ...)
- Model forms:
 - **global** model, probably with an SAR model to account for local correlation
 - **GWR** model: different coefficients of each predictor; different importance of predictors in different areas
- NY State example: importance/coefficient of home ownership to voting choice in clusters of urban vs. rural vs. suburban counties
 - could also account for this in a linear model with categorical predictor “county type” and interaction terms
 - but that would limit the spatial resolution to the county

- Prediction: ““Please also be aware that using GWR for prediction has no good basis anywhere for anything - and the standard errors should not be given any credibility. This is not what GWR is for at all.” – Roger Bivand¹
 - However, it is possible to predict with GWR by evaluating the local formula at each prediction point (not necessarily observation points)>
- Modelling: GWR does *not* account for **local spatial correlation** within each window; compare with GLS

¹<https://grokbase.com/t/r/r-sig-geo/106bgxksy4/gwr-analysis>

- **Spatial Autoregressive** (SAR) multiple regression models
 - account for local correlations to adjust global model coefficients, but still one model
- **Regression Kriging** (RK): the global trend is fit (multiple regression, SAR, random forests ...) and then adjusted locally by kriging the **residuals** and adding them to the trend prediction.
 - Assumes that the global trend is correct, but affected by local factors.
- **Kriging with External Drift** (KED) in a restricted neighbourhood
 - the trend is re-fit at each prediction point according to some restricted radius;
 - the residuals from this local trend, in the same neighbourhood are at the same time kriged;
 - uses a **model of spatial dependence** (variogram of the residuals)

GWR uses the normal OLS formulation:

- model: $y_i = \beta_0 + \sum_k \beta_k x_{ik} + \varepsilon_i$
- fit from sets of known $(y_i, \mathbf{x}_i)$
- the errors ε_i are I.I.D. and not spatially-correlated
- solution:

$$\hat{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

In a **global** model, all observations participate equally in a single model.

Note that GWR does *not* use Generalized Least Squares (GLS), no accounting for eventual spatial correlation of residuals.

OLS but in a **moving window**:

- coordinates of the central point (u_i, v_i) , e.g., (E, N) .
- **model:** $y_i = \beta_0(u_i, v_i) + \sum_k \beta_k(u_i, v_i)x_{ik} + \varepsilon_i$
 - again the errors are I.I.D. and *not* spatially-correlated
- The coefficient vary with the coordinates (u_i, v_i) , so are **re-fit at every fit point**.
- **Solution:**

$$\hat{\beta}(u_i, v_i) = (\mathbf{X}^T \mathbf{W}(u_i, v_i) \mathbf{X})^{-1} \mathbf{X}^T \mathbf{W}(u_i, v_i) \mathbf{y}$$

- $\mathbf{W}(\mathbf{u}_i, \mathbf{v}_i)$ are the weights of the points in the neighbourhood to be used to fit the regression
 - this is a **diagonal** matrix, *no correlation* between weights (compare GLS)
 - All observations are considered but some may have 0 weight.

- GWR is a **weighted least-squares** regression (WLS);
 - WLS: weight some observations more than others in computing the regression coefficients
 - example: inverse weight by measurement variance, gives more weight to more reliable observations
- the weights are chosen to represent the neighbourhood;
- the weights change at each point

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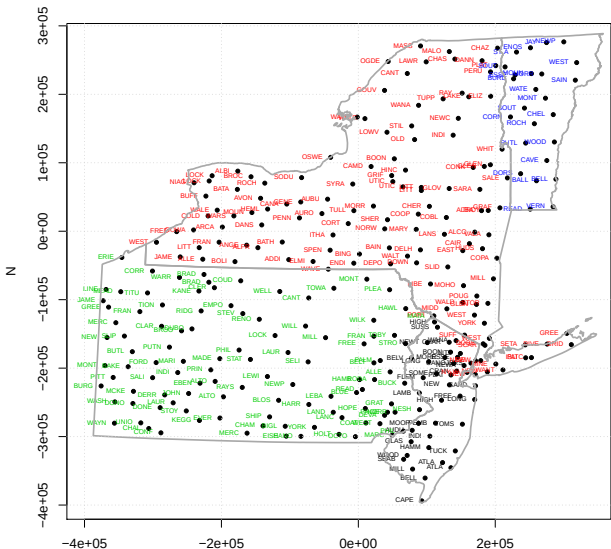
GWR Example 2 –
Georgia (USA)
poverty

Extensions to
GWR

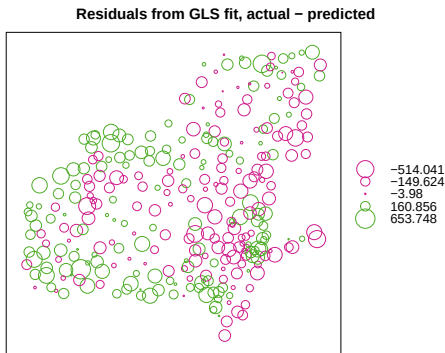
References

spgwr Bivand [1]; one of the authors of the sp package
GWmodel Gollini et al. [5]; Lu et al. [7]

- Four US States: VT, NY, NJ, PA
- 305 climate stations
- target variable: Growing Degree Days base-50° C (accumulated heat units for crop growth)
- predictors: North, East, elevation (square root)



- GLS: $\text{ANN_GDD50} \sim \text{sqrt}(\text{ELEVATION_}) + N$
- Fitted coefficients:
 - (Intercept) 3136.37
 - $\text{sqrt}(\text{ELEVATION_})$ -3.00 (per $\sqrt{m}$)
 - N -1.91 (per km)
 - E (not used, does not improve adjusted R^2)
 - spatial correlation of residuals effective range ≈ 52 km
- adjusted $R^2 \approx 0.86$, RMSE 217 GDD_50
- Interpretation: strong **regional** effect of elevation and Northing on the annual heat units
Easting not significant in the *global* (regional) model



Model was not equally good everywhere! And there are clear clusters of +/- residuals.

What to do about this model?

The model is successful over the region, but there are important local variations.

What to do?

- 1 **Krige the residuals** and add to the GLS prediction (GLS-RK)
 - This accounts for a **local** process, **within** the **regional** process
 - e.g., presence of large water bodies
- 2 **GWR** to fit the model locally
 - Will **miss the regional variation**
 - Assumes the process is **local**
 - Maybe will better fit locally, and reveal the local importance of the three predictors

Question: which seems more appropriate in this case?

GWR model – select a bandwidth

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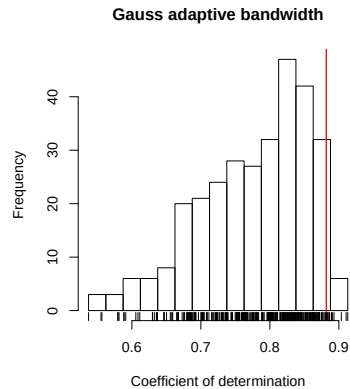
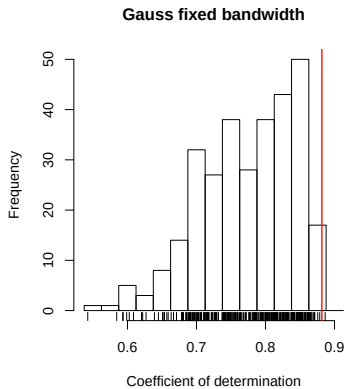
References

Use a Gaussian kernel; optimize by cross-validation

fixed 72.4 km

- at this radius a point receives $e^{1/2} = 0.6065$ weight.
- all points will be considered

adaptive 3.35% of the stations in each window, i.e., about 10 stations for each regression



Regional value shown with red vertical line
Most **local** models have a poorer fit
Because of the restricted range of predictors in a local window

GWR model – intercepts - feature-space distribution

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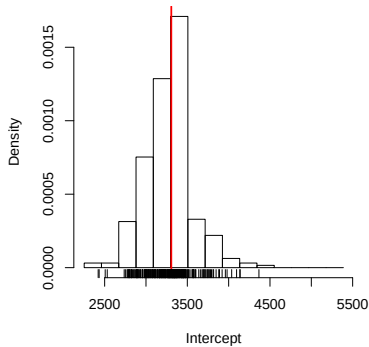
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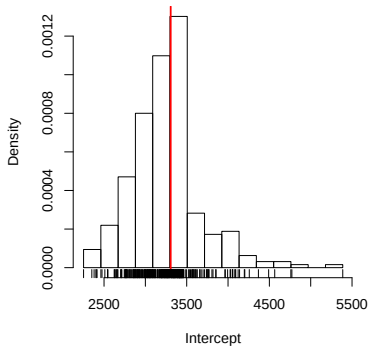
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Gauss fixed bandwidth



Gauss adaptive bandwidth



GWR model – intercepts - spatial distribution

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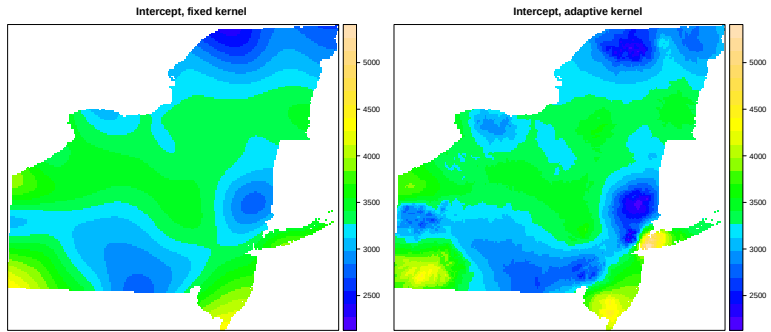
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Not the average! A centering constant. Note low values in southcentral PA & the Taconics as well as northern NY/VT

GWR model – elevation - feature-space distribution

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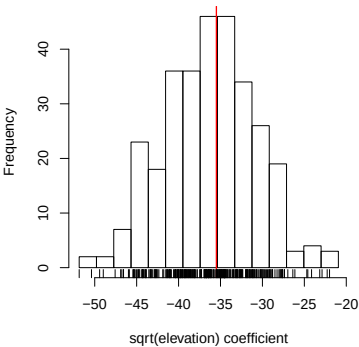
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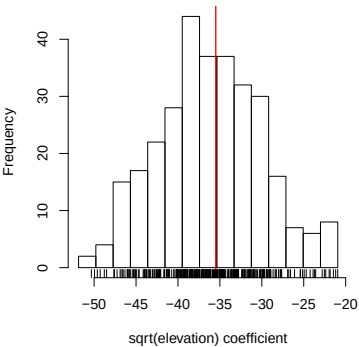
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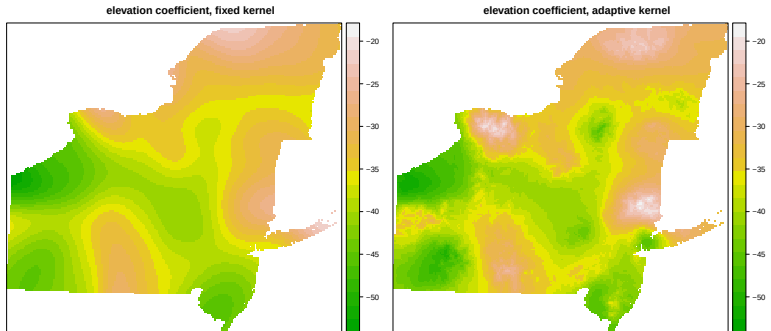
Gauss fixed bandwidth



Gauss adaptive bandwidth



GWR model – elevation - spatial distribution



Much of this pattern seems to be an artefact of GWR

GWR model – Northing - feature-space distribution

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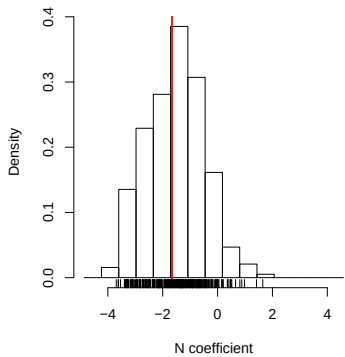
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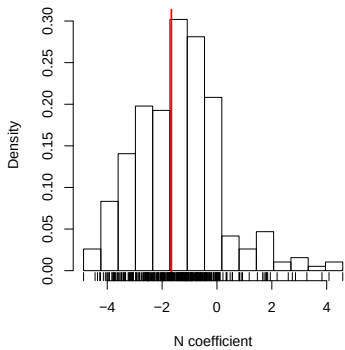
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GWR model – Northing - spatial distribution

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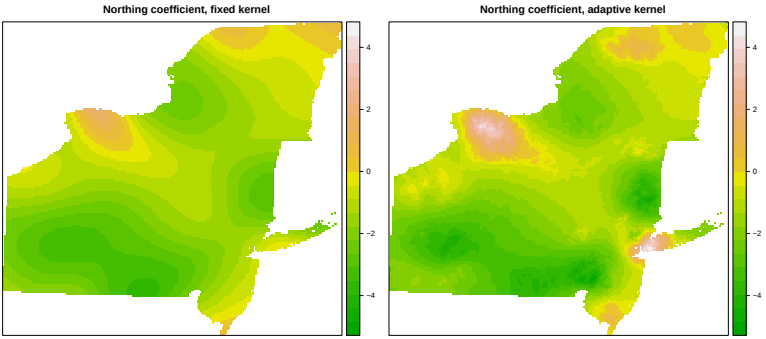
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Can be locally **positive**, disagrees with physical principles

GWR model – Easting - feature-space distribution

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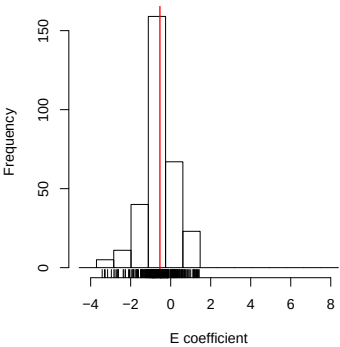
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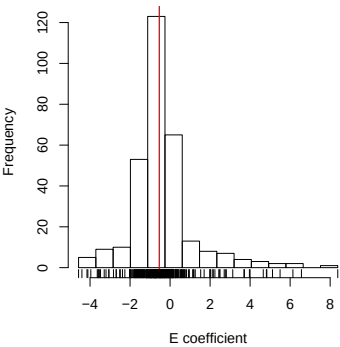
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GWR model – Easting - spatial distribution

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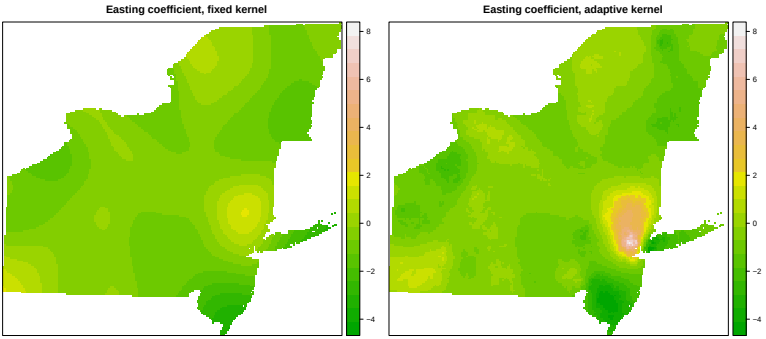
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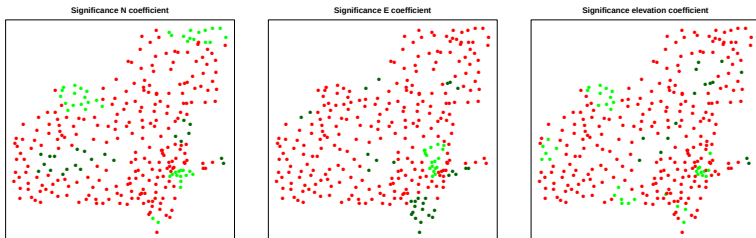
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Local effect in lower Hudson valley

Significance of coefficients



red: non-significant; dark green: negative; light green: positive

Intercepts are always highly significant, i.e., $\neq 0$; they centre the local regression

Interpretation: most local models are fit only with the local average (intercept)!

- **Global** model finds the average effect, over the *entire region*, of the predictors
 - the physically-plausible Northing and elevation are highly significant
 - these have a **wide range** of values over the region
 - good fit, over 85% of variance explained
- **GWR** model:
 - local models with an effective radius ≈ 100 km
 - wide range intercepts (averages) $\rightarrow$ local means
 - this takes out most of the effect of Northing
 - some effect of Northing, Easting near water bodies
 - elevation only important in windows with significant relief
 - usually much lower R^2 , less of each window is explained by factors other than the local mean
- **In this case** the GWR model is *not* justified.

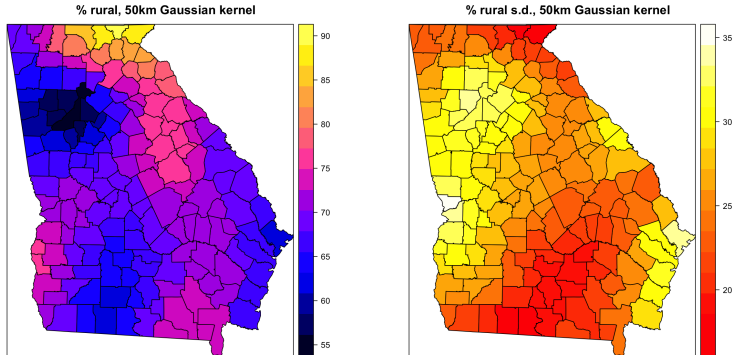
Example – Georgia (USA) poverty

- Georgia (USA) counties 1990 census; originally used in [3]
- Problem: how to explain the proportion of the population in **poverty**?
- Possible **predictors**: percent of population which is:
 - ① rural
 - ② has a bachelor's degree or higher
 - ③ elderly
 - ④ foreign-born
 - ⑤ African-American
- Practical application: if we know what is correlated with poverty (positive or negative), we can think of interventions

```
## lm(formula = lm.formula, data = educ.spdf@data)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -7.8282 -2.8418 -0.2404  2.6184 17.4764
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  7.506033   2.325226   3.228 0.001525 **
## PctRural     -0.007883   0.015780  -0.500 0.618121
## PctBach      -0.293767   0.083418  -3.522 0.000566 ***
## PctEld       0.709494   0.126583   5.605 9.46e-08 ***
## PctFB        0.148516   0.366098   0.406 0.685549
## PctBlack     0.259411   0.019638  13.210 < 2e-16 ***
##
## Multiple R-squared:  0.7078, Adjusted R-squared:  0.6982
```

- about 70% of the variability in poverty is explained.
- The strongest predictors are **education** (moderately negative), **elderly** (strongly positive), **black** (moderately positive).
- Proportion of **rural** residents has almost no effect – but is this because we are mixing urban (Atlanta, Savannah) and rural areas?
- Proportion of **foreign-born** residents has almost no effect

A null model can be used to find locally-weighted statistics of a target variable.



global mean 70.18

global s.d. 27.1

Note: bounding box about 443 x 514 km

- GWR depends on the choice of kernel
 - ① functional form
 - ② bandwidth
 - ③ fixed vs. adaptive
- Next slides show the difference between kernels

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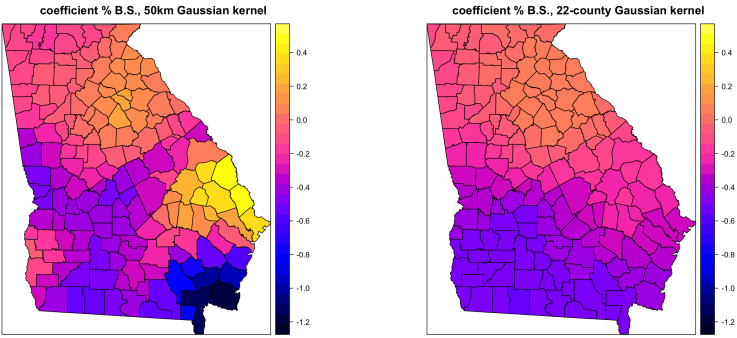
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global coefficient -0.29

Note: education is associated with *increased* poverty in E central (Athens – University of GeorgiaZA)

GWR coefficients – % elderly

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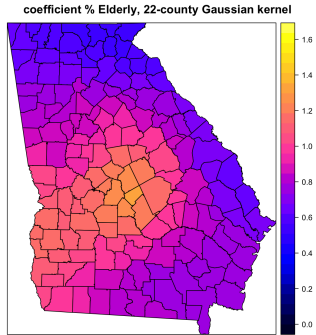
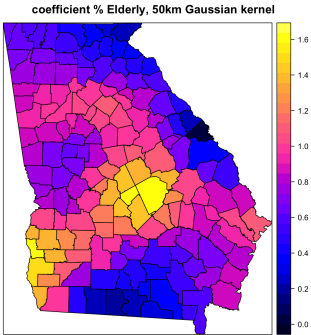
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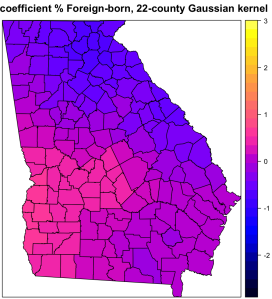
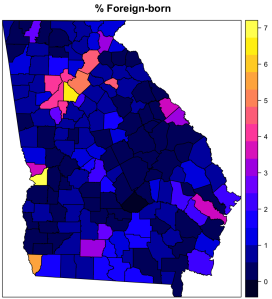
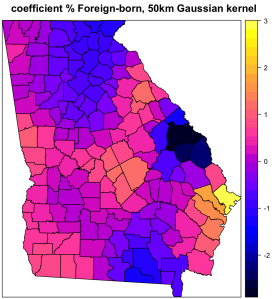
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Note the increased noise with the narrower kernel.

Artefacts – foreign-born



- Substantial differences in regression coefficients across map
- In some cases the sign changes
- Suggests **different causes/correlations** in different areas: look for local effects
- Substantial differences with choice of kernel
- **Question:** is 50 km with Gaussian weights an appropriate fixed bandwidth?
- **Question:** are 22 counties with Gaussian weights an appropriate adaptive bandwidth?

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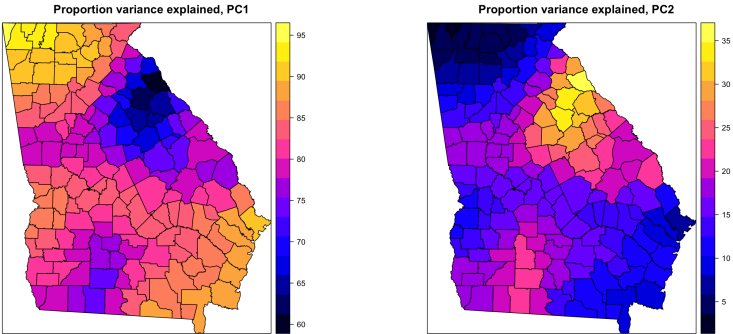
- GWR model with some coefficients **global**, i.e., *not* varying with the moving window
- Allows global/regional effects
 - Example: soil organic matter: affected by **regional** climate, but by **local** topographic effects [9]
- MGWR tests which predictors are fixed and which can vary (and at which bandwidth) [8]

Multiscale Geographically Weighted Regression (MGWR)

- Developed by Fotheringham et al. [4]
- GWR with **different bandwidths** for **different processes** (represented by predictors)
- computes an **optimal bandwidth vector** in which each element indicates the **spatial scale** at which a particular process takes place
- can interpret the various bandwidths to infer the spatial processes

- As with OLS regression, but now Principal Components [6]
- Look for the multivariate correlations among predictors in a moving window
- Interpret the PC loadings, per window
- Can use the PC scores to create new, uncorrelated variables

Georgia poverty predictors, 50 km Gaussian bandwidth



PC1 much more explanatory in NW GA, i.e., predictors are much more correlated there

- A useful tool to **investigate** spatial heterogeneity in regression models
 - changing coefficients, changing variable importance, changing R^2
 - the **bandwidth** reveals the **spatial scale** of the heterogeneity
- This should be **interpretable** in terms of the physical/social setting

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